

On a new type of the ℓ -adic regulator for algebraic number fields (The ℓ -adic regulator without logarithms)

L. V. Kuz'min*

Abstract

For an algebraic number field K such that prime ℓ splits completely in K we define a regulator $\mathfrak{R}_\ell(K) \in \mathbb{Z}_\ell$ that characterizes the subgroup of universal norms from the cyclotomic \mathbb{Z}_ℓ -extension of K in the completed group of S -units of K , where S consists of all prime divisors of ℓ . We prove that inequality $\mathfrak{R}_\ell(K) \neq 0$ follows from the ℓ -adic Schanuel conjecture and holds true for some Abelian extensions of imaginary quadratic fields.

1 Introduction

Let K be an algebraic number field and ℓ a fixed prime number. Then there is the ℓ -adic regulator $R_\ell(K) \in \mathbb{Q}_\ell$ of K in the sense of [1], which characterizes the ℓ -adic behavior of the group of units $U(K)$ of K . (Its definition is reproduced in Section 2, see Conjecture 3.) Nevertheless this regulator isn't perfect from the viewpoint of Iwasawa theory.

For example, suppose that prime ℓ splits completely in K . Let K_∞ be the cyclotomic \mathbb{Z}_ℓ -extension of K . In this case for any intermediate subfield K_n of the extension K_∞/K , where $[K_n : K] = \ell^n$ and $K = K_0$, one has a regulator $R_\ell(K_n)$, which characterizes the group of units $U(K_n)$ of K_n . As it was shown in [2], the behavior of the ℓ -adic exponents $\nu_\ell(R_\ell(K_n))$ for increasing n shows some interesting features. This event is connected with the

*This work was done with the financial support of the Russian Foundation for basic Research (grant no. 11-01-00588-a)

fact that the properties of the sequence of the groups of units $\{U(K_n)\}_{n \geq 0}$ reflect in a single object – a Galois module (with respect to the action of the Galois group $\Gamma := G(K_\infty/K)$) $\bar{U}(K_\infty) := \varprojlim (U(K_n)[\ell]/\mu_\ell(K_n))$, where $U(K_n)[\ell]$ is the pro- ℓ -completion of $U(K_n)$, $\mu_\ell(K_n)$ is the group of all roots of unity of ℓ -power degree in K_n and the projective limit is taken with respect to the norm maps. In this case for any n one has a natural projection $\pi_n : \bar{U}(K_\infty) \rightarrow U(K_n)[\ell]/\mu_\ell(K_n)$. In particular, one has a projection $\pi : \bar{U}(K_\infty) \rightarrow U(K)[\ell]/\mu_\ell(K)$. But if prime ℓ splits completely in K and Leopoldt conjecture holds for K and ℓ , then one has $\pi = 0$ (we should explain it more carefully in Section 2). So it is natural to consider instead of $U(K)[\ell]/\mu_\ell(K)$ some other object, which behaves better from the viewpoint of Iwasawa theory.

Namely, let S be the set of all places over ℓ in any field under consideration and $U_S(K_n)$ the group of S -units in K_n . Then one can link the sequence of groups $\{U_S(K_n)\}_{n \geq 0}$ to a single Γ -module $\bar{U}_S(K_\infty) := \varprojlim (U_S(K_n)[\ell]/\mu_\ell(K_n))$, where the limit is taken with respect to the norm maps. For any $n \geq 0$, there is a natural projection $\pi'_n : \bar{U}_S(K_\infty) \rightarrow U_S(K_n)[\ell]/\mu_\ell(K_n)$. The compactness of the groups $U_S(K_n)[\ell]/\mu_\ell(K_n)$ yields that the image of π'_n , which we denote by $U_{S,1}(K_n)$, coincides with the subgroup of universal norms in $U_S(K_n)[\ell]/\mu_\ell(K_n)$, that is,

$$U_{S,1}(K_n) = \bigcap_{m > n} N_{K_m/K_n}(U_S(K_m)[\ell]/\mu_\ell(K_m)). \quad (1.1)$$

The new ℓ -adic regulator, which we wish to define, should characterize the ℓ -adic behavior of the elements of $U_{S,1}(K_n)$. But in the present paper we shall treat only the case $n = 0$. Thus we shall define the new ℓ -adic regulator $\Re_\ell(K) \in \mathbb{Z}_\ell$ that characterize $U_{S,1}(K)$ under assumption that prime ℓ splits completely in K . (The case of arbitrary K will be treated in the next paper.) This restriction enable us to make the presentation more transparent.

Let $D = D(K)$ be a free additive \mathbb{Z}_ℓ -module generated by the elements of S . Then one can put to any $a \in U_S(K)[\ell]/\mu_\ell(K)$ its “principal divisor” $\text{div}(a) \in D$. Thus one gets a homomorphism $\text{div} : U_{S,1}(K) \rightarrow D$, which under assumption of validity of Leopoldt conjecture for K and ℓ , induces an injection $\text{div} : U_{S,1}(K) \hookrightarrow D$.

There is a standard \mathbb{Z}_ℓ -linear scalar product on D

$$\langle \ , \ \rangle : D \times D \rightarrow \mathbb{Z}_\ell. \quad (1.2)$$

Namely, if $S = \{v_1, \dots, v_m\}$ and $\alpha, \beta \in D$, $\alpha = \sum_{i=1}^m a_i v_i$, $\beta = \sum_{i=1}^m b_i v_i$, where $a_i, b_i \in \mathbb{Z}_\ell$, then we put $\langle \alpha, \beta \rangle = \sum_{i=1}^m a_i b_i$. Assuming the Leopoldt conjecture valid for K and ℓ , we prove that $U_{S,1}(K)$ is a free \mathbb{Z}_ℓ -module of rank $t = r_1 + r_2$, where r_1 and r_2 is the number of real and complex places of K respectively. Then we define a new regulator $\mathfrak{R}_\ell(K) \in \mathbb{Z}_\ell$ by the formula

$$\mathfrak{R}_\ell(K) = \det(\langle \operatorname{div}(e_i), \operatorname{div}(e_j) \rangle), \quad 1 \leq i, j \leq t,$$

where e_1, \dots, e_t is a \mathbb{Z}_ℓ -basis of $U_{S,1}(K)$.

Note that $\mathfrak{R}_\ell(K)$ is defined only up to multiplication by the square of an arbitrary element of \mathbb{Z}_ℓ^\times .

In the present paper we consider the following

The main conjecture. *Let K be an algebraic number field such that prime ℓ splits completely in K . Then $\mathfrak{R}_\ell(K) \neq 0$.*

Obviously, the main conjecture is equivalent to assertion that the product (1.2) induces a non-degenerate scalar product on $U_{S,1}(K)$, which is defined by

$$\{\alpha, \beta\} = \langle \operatorname{div}(\alpha), \operatorname{div}(\beta) \rangle \quad (1.3)$$

for any $\alpha, \beta \in U_{S,1}(K)$.

We prove that the main conjecture follows from the ℓ -adic Schanuel conjecture (see Conjecture 1 in Section 2). Then we give an unconditional proof of the main conjecture for some types of Abelian extensions of imaginary quadratic fields.

Now we shall give the plan of the paper and describe our main results in more details.

In Section 2 we remind some necessary definitions and formulate the conjectures that we will use in sequel. Some of them, such as Conjectures 1 and 2) are well-known, while Conjectures 3 and 4 were firstly formulated by the author in [1]. The main result of Section 2 is Theorem 1, which gives the \mathbb{Z}_ℓ -rank of $U_{S,1}(K)$. Moreover, if K/\mathbb{Q} is a Galois extension with Galois group G and such that the Leopoldt conjecture holds for a pair (K, ℓ) , then Theorem 1 asserts that the modules $(U(K)[\ell] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell$ and $U_{S,1}(K) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ are isomorphic over $\mathbb{Q}_\ell[G]$. (Here and in many cases later we write the multiplication additively).

If K/\mathbb{Q} is not Galois, then Theorem 1 asserts only the equality of dimensions of these two \mathbb{Q}_ℓ -spaces. The proof of this theorem depends heavily on class field theory and Iwasawa theory. We shall use also an important

consequence of Theorem 1 on an existence of a special element in the group $U_{S,1}(K)$, which is analogous to the Artin unit in the group of units $U(K)$. Note that we don't assume that ℓ splits completely in K .

In Section 3 we use this consequence of Theorem 1 to produce a conditional proof of the main conjecture. Namely, assuming that Conjecture 1 holds, we prove that $\Re_\ell(K) \neq 0$ for any algebraic number field K and any ℓ that splits completely in K . In this proof one can assume that K/\mathbb{Q} is Galois with Galois group G and K contains an imaginary quadratic field k . To do this, we introduce another conjecture (Conjecture 5), which is similar to Leopoldt conjecture. But in distinction to the latter, it is formulated in terms of a Galois module generated by $\alpha \in K$ such that its principal divisor (α) equals to $\mathfrak{l}^{\mathbf{h}}$, where \mathfrak{l} is a prime divisor of ℓ in K and \mathbf{h} is a natural number. (The class number of K , for example.) This conjecture is a consequence of Conjecture 1, but we hope that it may be more tractable in some cases.

Using Conjecture 5, man can define explicitly some group $U_{S,2}(K)$, which contains $U_{S,1}(K)$. The group $U_{S,2}(K)$ is the group of local universal norms in $U_S(K)[\ell]/\mu_\ell(K)$. Using Conjecture 1, man can prove that the index $(U_{S,2}(K) : U_{S,1}(K))$ is finite (Proposition 3.5). It should be mentioned that the last result gives a conditional proof of Conjecture 2 in [3] for the case, when ℓ splits completely in K .

During calculation of the group $U_{S,2}(K)$, there appears some system of coefficients $\{c_h\}_{h \in H}$, where $H = G(K/k)$ (see (3.11)), which enter the relations between the ℓ -adic logarithms of some Artin unit ε in $U(K)$ and its conjugate, and the ℓ -adic logarithms of $\sigma(\alpha)$, $\sigma \in G$. We prove modulo Conjecture 1 (Lemma 3.1), that after omitting from the system $\{c_h\}$ any element c_{h_0} , the rest elements form an algebraically independent set. Then we introduce an analog of the regulator $\Re_\ell(K)$, which characterizes not $U_{S,1}(K)$, but the group $U_{S,2}(K)$. We prove that this analog is a non-zero polynomial in $\{c_h\}_{h \in H, h \neq h_0}$. It proves Theorem 2, according to which $\Re_\ell(K) \neq 0$, if Conjecture 1 holds.

In Section 4 we give an unconditional proof of the main conjecture for one particular case (Theorem 3). We assume that K is Galois over \mathbb{Q} and Abelian over an imaginary quadratic field k . We assume also that the only non-trivial automorphism τ of k/\mathbb{Q} acts on the Galois group $H = G(k/\mathbb{Q})$ by inversion, i. e. $\tau h \tau^{-1} = h^{-1}$ for any $h \in H$. We assume also that any irreducible character χ of H with values in $\bar{\mathbb{Q}}_\ell$ is conjugated with χ^{-1} over \mathbb{Q}_ℓ .

The proof is based on the same idea as the proof of Theorem 3.1 in [1].

Namely, instead of the product (1.3) we consider some more complicated product (4.1), which has some additional symmetry (Proposition 4.1). If the latter product is non-generate, we get immediately that $\mathfrak{R}_\ell(K) \neq 0$, nevertheless we can state this only under additional assumption that χ and χ^{-1} are conjugate for any $\chi \in \hat{H}$.

The last condition depends only on the pair of numbers (ℓ, m) , where m is an exponent of H . We give a full list of all such pairs in Proposition 4.3. One can put a question in another way. Suppose K to be fixed. What is the set of all primes ℓ such that Theorem 3 holds for (K, ℓ) , that is, $\mathfrak{R}_\ell(K) \neq 0$? The answer is given in Proposition 4.4.

A part of this paper was written when the author was a guest of Hausdorff research institute for mathematics, Bonn (Trimester Program “Arithmetic and Geometry” 2013). The author would like to thank the collaborators of the institute for hospitality and the participants of the Trimester for stimulating communion.

2 Preliminary results

Let $\bar{\mathbb{Q}}_\ell$ be the algebraic closure of the rational ℓ -adic number field \mathbb{Q}_ℓ and $\log : \bar{\mathbb{Q}}_\ell^\times \rightarrow \bar{\mathbb{Q}}_\ell$ the ℓ -adic logarithm in the sense of Iwasawa. Note that in this paper we use neither real nor complex logarithms. So the denotation \log for the ℓ -adic logarithm would not cause any ambiguity. Thus, \log is a homomorphism of the multiplicative group $\bar{\mathbb{Q}}_\ell^\times$ into the additive group $\bar{\mathbb{Q}}_\ell$. This homomorphism is uniquely defined by conditions $\log(\ell) = 0$, $\log(\xi) = 0$ for any root of unity ξ , whereas \log is defined on the group of principal units $U^{(1)}(\bar{\mathbb{Q}}_\ell)$ by the usual power series.

A well-known ℓ -adic Schanuel conjecture asserts that for any $x_1, \dots, x_n \in \bar{\mathbb{Q}}_\ell^\times$ such that ℓ, x_1, \dots, x_n are multiplicatively independent, that is the equality $\ell^c \prod_{i=1}^n x_i^{c_i} = 1$ does not take place for any non-zero vector $(c, c_1, \dots, c_n) \in \mathbb{Z}^{n+1}$, the transcendence degree of $\mathbb{Q}(x_1, \dots, x_n, \log x_1, \dots, \log x_n)$ over \mathbb{Q} is at least n . In the present paper we shall use only the following particular case of this conjecture.

Conjecture 1. *Let $x_1, \dots, x_n \in \bar{\mathbb{Q}}_\ell^\times$ be algebraic over \mathbb{Q} and ℓ, x_1, \dots, x_n are multiplicatively independent. Then $\log x_1, \dots, \log x_n$ are algebraically independent over \mathbb{Q} .*

Let K be an algebraic number field and ℓ a fixed prime number. Then

there is a natural injection $i : K \hookrightarrow K \times_{\mathbb{Q}} \mathbb{Q}_{\ell} \cong \prod_{i=1}^m K_{v_i}$, where K_{v_i} is the completion of K with respect to v_i and v_1, \dots, v_m run all the places of K over ℓ (here we do not assume that ℓ splits completely in K).

Considering K_{v_i} as subfields of $\bar{\mathbb{Q}}_{\ell}$, we obtain injections $\sigma_1, \dots, \sigma_m$ of K into $\bar{\mathbb{Q}}_{\ell}$ such that $\sigma_i(K) \subset K_{v_i}$ and $i(a) = \{\sigma_1(a), \dots, \sigma_m(a)\} \in \prod_{i=1}^m K_{v_i}$ for any $a \in K$. In particular, to any $x \in K^{\times}$ one can consider a vector

$$\log_{\ell}(x) = \{\log \sigma_1(x), \dots, \log \sigma_m(x)\}. \quad (2.1)$$

Thus, we get a homomorphism

$$\log_{\ell} : K^{\times} \rightarrow \prod_{i=1}^m K_{v_i}.$$

If K is a Galois extension of \mathbb{Q} with the Galois group G , then the last homomorphism is a G -homomorphism. If ℓ splits completely in K , one has $m = [K : \mathbb{Q}]$ and $\prod_{i=1}^m K_{v_i} \cong \mathbb{Q}_{\ell}^m$.

Conjecture 2. (*Leopoldt conjecture*). *Let $u_1, \dots, u_r \in U(K)$ be multiplicatively independent units of K . Then $i(u_1), \dots, i(u_r)$ are multiplicatively independent over \mathbb{Z}_{ℓ} in $\prod_{i=1}^m K_{v_i}^{\times}$. In other words, the vectors $\log_{\ell} u_1, \dots, \log_{\ell} u_r$ are linearly independent over \mathbb{Q}_{ℓ} .*

The injection $i : U(K) \hookrightarrow \prod_{i=1}^m U(K_{v_i})$, where $U(K_{v_i})$ is the group of units of K_{v_i} , induces a homomorphism $i' : U(K)[\ell] \rightarrow \prod_{i=1}^m U(K_{v_i})$, where $U(K)[\ell]$ is the pro- ℓ -completion of $U(K)$. Conjecture 2 is equivalent to the assertion that i' is a monomorphism. In general case, the group $E_{\ell}(K) := \ker i'$ is known as Leopoldt kernel. So Leopoldt conjecture asserts that $E_{\ell}(K) = 1$ for any (K, ℓ) . It is well known that Leopoldt conjecture follows from the conjecture 1. Leopoldt conjecture is proved for Abelian fields and Abelian extensions of imaginary quadratic fields.

For an algebraic number field K and a prime number ℓ , let K_{∞} be the cyclotomic \mathbb{Z}_{ℓ} -extension of K , $G(K_{\infty}/K) := \Gamma \cong \mathbb{Z}_{\ell}$ the Galois group of K_{∞}/K and γ a fixed topological generator of Γ . By K_n we denote the unique intermediate subfield of K_{∞}/K such that $[K_n : K] = \ell^n$. Hence, $K = K_0$. We shall need some results on the group $E_{\ell}(K)$ and the sequence of groups $E_{\ell}(K_n)$. The next statement, which is in fact a theorem, is known as feeble Leopoldt conjecture.

Proposition 2.1. *(See, for example, [3]). The groups $E(K_n)$ stabilize since some n , that is, there is an index n , which depends only on K and ℓ , such that the natural injection $E_\ell(K_n) \hookrightarrow E_\ell(K_{n_1})$ is an isomorphism for any $n_1 > n$.*

Let M be the maximal Abelian ℓ -extension of K_∞ unramified outside ℓ and $X = G(M/K_\infty)$. Then Γ acts by conjugation on X , and thus X becomes a finitely generated Λ -module, where $\Lambda := \mathbb{Z}_\ell[[\Gamma]]$ is a completed group ring of Γ .

Proposition 2.2. *([3], Proposition 5.4). There exists a natural isomorphism $X^\Gamma \cong E_\ell(K)$.*

The next conjecture is a strengthening of Leopoldt conjecture. It is a consequence of Conjecture 1 (see [1, Appendix]). We formulate it, since it motivate many of our later considerations.

Conjecture 3. *(Conjecture on the ℓ -adic regulator [1]). For an algebraic number field K and prime ℓ , let $\varepsilon_1, \dots, \varepsilon_r$ be a system of fundamental units of K and $\varepsilon_0 = 1 + \ell$ ($\varepsilon_0 = 5$ if $\ell = 2$). Put $R_\ell(K) := \det(\mathrm{Sp}_{K/\mathbb{Q}}(\log_\ell \varepsilon_i \cdot \log_\ell \varepsilon_j))$, $0 \leq i, j \leq r$, where by $\mathrm{Sp}_{K/\mathbb{Q}}$ we denote the mapping $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell$ induced by the trace map $\mathrm{Sp}_{K/\mathbb{Q}}: K \rightarrow \mathbb{Q}$. Then $R_\ell(K) \neq 0$ for any K and ℓ .*

A relation of the following conjecture to Conjecture 3 is the same as that of the feeble Leopoldt conjecture and Leopoldt conjecture. But in distinction with the feeble Leopoldt conjecture, the conjecture, which we formulate below, is stated only in some particular cases. To formulate it, we need a notion of the relative ℓ -adic regulator. Let K/k be an extension of algebraic number fields and $U(K/k)$ the group of relative units of K over k , that is,

$$U(K/k) = \{ x \in U(K) \mid N_{K/k}(x) \in \mu(k) \},$$

where $\mu(k)$ is the group of all roots of unity in k , and u_1, \dots, u_t some system of fundamental units of $U(K/k)$. It means that u_1, \dots, u_t are independent and generate with the group $\mu(K)$ all the group $U(K/k)$. Then the relative ℓ -adic regulator $R_\ell(K/k)$ is defined by the formula

$$R_\ell(K/k) = \det(\mathrm{Sp}_{K/\mathbb{Q}}(\log_\ell u_i \log_\ell u_j)), \quad 1 \leq i, j \leq t.$$

Conjecture 4. (*Feeble conjecture on the ℓ -adic regulator [1]*). For an algebraic number field K and its cyclotomic \mathbb{Z}_ℓ -extension K_∞ , there is an index n_0 , which depends only on K and ℓ , such that $R_\ell(K_m/K_n) \neq 0$ for any $m > n \geq n_0$.

The regulator $\mathfrak{R}_\ell(K)$, which we determine and learn in the present paper, characterize the group of universal global norms $U_{S,1}(K)$ defined in (1.1). So we need a characterization of this group as Galois module. In the next theorem we put no restrictions on decomposition type of prime ℓ in K .

Theorem 1. *Let K be a finite Galois extension of \mathbb{Q} with a Galois group G and $R = \mathbb{Q}_\ell[G]$. Suppose that the Leopoldt conjecture holds for (K, ℓ) . Then there is an isomorphism of R -modules*

$$\varphi: U_{S,1}(K) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong (U(K) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell$$

(here we write the group operation in $U_{S,1}(K)$ and $U(K)$ additively). Without assumption that K is Galois over \mathbb{Q} , the module $U_{S,1}(K)$ has \mathbb{Z}_ℓ -rank $r_1 + r_2$, where r_1 and r_2 is the number of real and complex places of K respectively.

Proof. We shall state only the first part of the theorem. To state the second one, man has to repeat the proof of the first part substituting everywhere isomorphisms of R -modules by isomorphisms of corresponding \mathbb{Q}_ℓ -spaces.

It follows from the representation theory of finite groups, that any finitely generated R -module may be uniquely presented as a direct sum of irreducible R -modules. In particular, for R -modules A, B, C the condition $A \oplus B \cong A \oplus C$ yields $B \cong C$.

Let M be the maximal Abelian ℓ -extension of K_∞ unramified outside ℓ and $X = G(M/K_\infty)$. Let M_n be the maximal Abelian ℓ -extension of K_n unramified outside ℓ . Then for any n there are inclusions $K \subseteq K_n \subset K_\infty \subset M_n \subseteq M$, and the Galois group $G(M_n/K_n)$ enters the exact sequence

$$1 \longrightarrow X_n \longrightarrow G(M_n/K_n) \longrightarrow \Gamma_n \longrightarrow 1,$$

where $X_n = G(M_n/K_\infty) = X/(\gamma_n - 1)X$, $\gamma_n = \gamma^{\ell^n}$ and $\Gamma_n = G(K_\infty/K_n) = \langle \gamma_n \rangle$. In particular, for $n = 0$ one gets an exact sequence of $\mathbb{Z}_\ell[G]$ -modules

$$1 \longrightarrow X_0 \longrightarrow G(M_0/K) \longrightarrow \Gamma \longrightarrow 1. \quad (2.2)$$

To prove the theorem, we shall determine the R -module $Y(K) := G(M_0/K) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ in two different ways.

Let $V(K)$ be a subgroup of $G(M_0/K)$ generated by the inertia subgroups for all v over ℓ . Then the G -module $V(K)$ is of finite index in $G(M_0/K)$, and, according to global class field theory, $V(K)$ contains in the exact sequence of G -modules

$$1 \longrightarrow U(K)[\ell] \xrightarrow{\alpha} \prod_{v|\ell} U(K_v)_\ell \longrightarrow V(K) \longrightarrow 1, \quad (2.3)$$

where $U(K_v)$ is the group of units of the local field K_v , $U(K_v)_\ell$ is the ℓ -component of $U(K_v)$ (thus it coincides with the group of principal units $U^{(1)}(K_v)$), and the product is taken over all places v of K over ℓ . Note that α is an injection, since we assume that the Leopoldt conjecture holds for (K, ℓ) .

Since the ℓ -adic logarithm maps $U^{(1)}(K_v)$ on some full lattice in K_v and has finite kernel, it induces an isomorphism of R -modules

$$\prod_{v|\ell} (U(K_v)_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \cong \prod_{v|\ell} K_v \cong K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong R, \quad (2.4)$$

where the last isomorphism follows from the theorem on normal base. Thus the exact sequence (2.3) yields an isomorphism

$$(U(K)[\ell] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \oplus Y(K) \cong R, \quad (2.5)$$

where $Y(K) := V(K) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Now we shall characterize the G -module $Y(K)$ in another way. To do this, we calculate the G -module $X_0 = G(M_0/K_\infty)$ in terms of Iwasawa theory. Note that $G(M_n/K_n)$ enters the exact sequence of Γ/Γ_n -modules

$$1 \longrightarrow W(K_n) \longrightarrow G(M_n/K_n) \longrightarrow C(K_n) \longrightarrow 1, \quad (2.6)$$

where $W(K_n)$ is a subgroup of $G(M_n/K_n)$ generated by the decomposition subgroups of all places v over ℓ and $C(K_n)$ is a Galois group of the maximal Abelian unramified ℓ -extension of K_n , in which all places $v|\ell$ splits completely. Note that by global class field theory $C(K_n)$ is canonically isomorphic to $\text{Cl}'_\ell(K_n)$, where $\text{Cl}'_\ell(K_n)$ is a factor group of the ℓ -component $\text{Cl}_\ell(K_n)$ of the class group $\text{Cl}(K_n)$ of K_n by a subgroup generated by all prime divisors of ℓ .

For any pair of indices $m > n$ man can include the sequences (2.6) for K_n and K_m into the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & W(K_m) & \longrightarrow & G(M_m/K_m) & \longrightarrow & C(K_m) \longrightarrow 1 \\ & & \downarrow f_{m,n} & & \downarrow & & \downarrow \\ 1 & \longrightarrow & W(K_n) & \longrightarrow & G(M_n/K_n) & \longrightarrow & C(K_n) \longrightarrow 1 \end{array} \quad (2.7)$$

The vertical arrows of this diagram are induced by the restriction of automorphisms.

Passing to projective limit in (2.6) with respect to the vertical arrows, we obtain an exact sequence of Γ -modules

$$1 \longrightarrow W(K_\infty) \longrightarrow X \longrightarrow T_\ell(K_\infty) \longrightarrow 1, \quad (2.8)$$

where $W(K_\infty)$ is a subgroup of X generated by decomposition subgroups of all v over ℓ and $T_\ell(K_\infty)$ a Galois group of the maximal Abelian unramified ℓ -extension of K_∞ , in which all places of K_∞ over ℓ split completely.

By Proposition 2.2 one has $X^\Gamma = 1$, hence (2.8) yields an exact homological sequence

$$1 = X^\Gamma \longrightarrow T_\ell(K_\infty)^\Gamma \longrightarrow W(K_\infty)_0 \longrightarrow X_0 \longrightarrow T_\ell(K_\infty)_0 \longrightarrow 1, \quad (2.9)$$

where for a Γ -module A we denote by A_0 the group $A/(\gamma - 1)A$.

By Iwasawa theory $T_\ell(K_\infty) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a finite-dimensional \mathbb{Q}_ℓ -space.

Lemma 2.1. *Let K be a finite Galois extension of \mathbb{Q} with Galois group G and $R = \mathbb{Q}_\ell[G]$. Let A be a finite-dimensional \mathbb{Q}_ℓ -space with continuous action of the Galois group $G(K_\infty/\mathbb{Q})$. Then the R -modules A^Γ and $A_0 = A/(\gamma - 1)A$ are isomorphic.*

Proof. We use induction by dimension of A . If $\dim A = 1$, then the statement of the lemma is obvious. Let $\dim A = d$ and assume that the lemma holds for all dimensions less than d . If $A^\Gamma = 0$, then multiplication by $(\gamma - 1)$ is an injection and thus an isomorphism. So $A_0 = 0$ and the lemma holds. Suppose that $A^\Gamma \neq 0$. Put $B = A^\Gamma \cap (\gamma - 1)A$. If $B = 0$ then, taking into account that $\dim A^\Gamma = \dim A_0$, we see that the natural surjection $A \rightarrow A_0$ induces an isomorphism of R -modules $A^\Gamma \rightarrow A_0$. If $B \neq 0$ then we put $B_1 = \{x \in A/B \mid (\gamma - 1)\bar{x} \in B\} = (A/B)^\Gamma$, where \bar{x} is some pre-image of x in A . Then multiplication by $\gamma - 1$ defines an R -surjection $B_1 \rightarrow B$, whose kernel coincides with A^Γ/B . Therefore, $B_1 \cong A^\Gamma$ as R -modules, that is, the R -modules A^Γ and $(A/B)^\Gamma$ are isomorphic. Since $\dim B \neq 0$, we have $(A/B)^\Gamma \cong (A/B)_0$ by assumption of induction but $(A/B)_0 \cong A_0$. This proves the lemma.

Taking tensor multiplication of (2.9) by \mathbb{Q}_ℓ and applying Lemma 2.1 to the R -module $T_\ell(K_\infty) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, we obtain

$$W(K_\infty)_0 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong X_0 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \quad (2.10)$$

By global class field theory for any $n \geq 0$ the group $W(K_n)$ enters the exact sequence of $G(K_n/\mathbb{Q})$ -modules

$$1 \longrightarrow E(K_n) \longrightarrow U_S(K_n)[\ell] \longrightarrow \prod_{v|\ell} (K_{n,v}^\times[\ell]) \longrightarrow W(K_n) \longrightarrow 1, \quad (2.11)$$

where $[\ell]$ means pro- ℓ -completion. If $m > n$ then man can include the sequences (2.11) for K_m and K_n in the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E_\ell(K_m) & \longrightarrow & U_S(K_m)[\ell] & \longrightarrow & \prod_{v|\ell} (K_{m,v}^\times[\ell]) & \longrightarrow & W(K_m) & \longrightarrow & 1 \\ & & \downarrow N_{m,n} & & \downarrow N_{m,n} & & \downarrow N_{m,n} & & \downarrow f_{m,n} & & \\ 1 & \longrightarrow & E_\ell(K_n) & \longrightarrow & U_S(K_n)[\ell] & \longrightarrow & \prod_{v|\ell} (K_{n,v}^\times[\ell]) & \longrightarrow & W(K_n) & \longrightarrow & 1 \end{array}$$

where the maps $N_{m,n}$ are induced by the norm maps from K_m into K_n and $f_{m,n}$ is the map of the diagram (2.7).

Passing in (2.11) to projective limit with respect to the vertical maps of preceding diagram and taking into account that $\varprojlim E_\ell(K_n) = 1$ by Proposition 2.1, we get an exact sequence of Γ -modules

$$1 \longrightarrow U_S(K_\infty) \longrightarrow \varprojlim \prod_{v|\ell} (K_{n,v}^\times[\ell]) \longrightarrow W(K_\infty) \longrightarrow 1, \quad (2.12)$$

where $U_S(K_\infty) = \varprojlim U_S(K_n)[\ell]$ and the limit is taken with respect to the norm maps. Denote the group $\varprojlim \prod_{v|\ell} (K_{n,v}^\times[\ell])$ by $\tilde{H}(K_\infty)$. Since $W(K_\infty) \subseteq X$, it follows from Proposition 2.2 that $W(K_\infty)^\Gamma = 0$, therefore, (2.12) induces an exact homological sequence

$$1 \longrightarrow U_S(K_\infty)_0 \longrightarrow \tilde{H}(K_\infty)_0 \longrightarrow W(K_\infty)_0 \longrightarrow 1. \quad (2.13)$$

Suppose that for any field K_n , where $n \geq 0$ one has a Galois \mathbb{Z}_ℓ -module $A(K_n)$. Moreover, suppose that for any pair of indices $m > n$ one has an injection $i(n, m): A(K_n) \rightarrow A(K_m)$ such that $i(n, m)(A(K_n)) = A(K_m)^{\Gamma_n}$, and for any triple of $m > n > k$ one has $i(k, m) = i(n, m) \circ i(k, n)$. Then for any $m > n$ the norm map $N_{\Gamma_n/\Gamma_m}: A(K_m) \rightarrow A(K_m)^{\Gamma_n}$ with respect to the Γ_n/Γ_m induces a map $N_{m,n}: A(K_m) \rightarrow A(K_n)$, $N_{m,n} = i(n, m)^{-1} \circ N_{\Gamma_n/\Gamma_m}$. Put $A(K_\infty) = \varprojlim A(K_n)$, where the limit is taken with respect to the maps $N_{m,n}$. Then the collection of the maps $N_{m,0}: A(K_m) \rightarrow A(K)$ defines a map $N: A(K_\infty) \rightarrow A(K)$. On the other hand, any map $N_{m,n}$ permits passing

through the natural projection $\pi_m: A(K_m) \rightarrow A(K_m)/(\gamma - 1)A(K_m)$. Thus, the map N permits passing through the natural projection $\pi: A(K_\infty) \rightarrow A(K_\infty)_0 = A(K_\infty)/(\gamma - 1)A(K_\infty)$, that is, there exists a map $j: A(K_\infty)_0 \rightarrow A(K)$ such that $N = j \circ \pi$. We shall deal with two such collections of Galois modules: $A_1(K_n) = U_S(K_n)[\ell]$ and $A_2(K_n) = \prod_{v|\ell} (K_{n,v}^\times[\ell])$. Denoting the maps N, π and j for these collections by N_1, π_1, j_1 and N_2, π_2, j_2 respectively, we see that man can include the exact sequences (2.12) and (2.13) into a commutative diagram with exact lines, where $j_1 \circ \pi_1 = N_1$ and $j_2 \circ \pi_2 = N_2$,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_S(K_\infty) & \longrightarrow & \tilde{H}(K_\infty) & \longrightarrow & W(K_\infty) & \longrightarrow & 1 \\
& & \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 & & \\
1 & \longrightarrow & U_S(K_\infty)_0 & \xrightarrow{\alpha} & \tilde{H}(K_\infty)_0 & \longrightarrow & W(K_\infty)_0 & \longrightarrow & 1 \quad (2.14) \\
& & \downarrow j_1 & & \downarrow j_2 & & & & \\
& & U_S(K)[\ell] & \xrightarrow{\beta} & \prod_{v|\ell} (K_v^\times[\ell]) & & & &
\end{array}$$

Lemma 2.2. *The homomorphisms j_1 and j_2 are injections.*

Proof. At first, we prove that j_2 is an injection. To do this, it is enough to check that $\hat{H}^{-1}(\Gamma/\Gamma_n, \prod_{v|\ell} (K_{n,v}^\times[\ell])) = 0$ for any n , where \hat{H}^{-1} is the Tate group of cohomologies. Since K is normal over \mathbb{Q} , there is an index $r \geq 0$ such that all places of K over ℓ split completely in K_r/K and any place $v|\ell$ of K_r has the unique extension to K_∞ . If $n \leq r$ then the Γ/Γ_n -module $D_n := \prod_{v|\ell} (K_{n,v}^\times[\ell])$ is induced, hence cohomologically trivial. If $n > r$ then D_n is relatively induced. Therefore by the Shapiro lemma one has $\hat{H}^{-1}(\Gamma/\Gamma_n, D_n) = \hat{H}^{-1}(\Gamma_r/\Gamma_n, \prod_{v \in S_0} (K_{n,v}^\times[\ell])) = \prod_{v \in S_0} \hat{H}^{-1}(\Gamma_r/\Gamma_n, K_{n,v}^\times[\ell])$, where S_0 is a set of places of K_n , which for any place $v_0|\ell$ of K contains exactly one its extension to K_n . The last group is zero by virtue of Gilbert's Theorem 90, hence $\hat{H}^{-1}(\Gamma/\Gamma_n, D_n) = 0$. It means that π_2 and N_2 have the same kernel, hence j_2 is an injection. Thus, j_2 settles an isomorphism between $\tilde{H}(K_\infty)_0$ and the subgroup of (local) universal norms in $\prod_{v|\ell} (K_v^\times[\ell])$.

Since $j_2 \circ \alpha = \beta \circ j_1$ and α, β, j_2 are injections, we obtain that j_1 is also an injection. This proves the lemma.

Thus, j_1 settles an isomorphism between $U_S(K_\infty)_0$ and the group of global universal norms $\text{Im} N_1 = U_{S,1}(K)$.

For a finite extension L of \mathbb{Q}_ℓ , let $F(L)$ be the subgroup of all universal norms in $L^\times[\ell]$ from L_∞ . In other words, $F(L) = \cap_{n=1}^\infty N_{L_n/L}(L_n^\times[\ell])$. Then,

identifying in the second line of (2.14) the modules $U_S(K_\infty)_0$ and $\tilde{H}(K_\infty)_0$ with $U_{S,1}(K)$ and $\prod_{v|\ell} F(K_v)$ (the identification is given via the maps j_1 and j_2 respectively), we get an exact sequence

$$1 \longrightarrow U_{S,1}(K) \longrightarrow \prod_{v|\ell} F(K_v) \longrightarrow W(K_\infty)_0 \longrightarrow 1. \quad (2.15)$$

Lemma 2.3. *Let K be a Galois extension of \mathbb{Q} with Galois group G and $R = \mathbb{Q}_\ell[G]$. Then there is an isomorphism of R -modules*

$$\left(\prod_{v|\ell} F(K_v) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong R.$$

Proof. Let L be a finite Galois extension of \mathbb{Q}_ℓ with Galois group H and $\theta_L: L^\times[\ell] \hookrightarrow G_{\ell,L}^{ab}$ the reciprocity map, where $G_{\ell,L}^{ab}$ is a Galois group of the maximal Abelian ℓ -extension of L . Let $\pi: G_{\ell,L}^{ab} \rightarrow \Gamma$ be the natural projection, where Γ is a Galois group of the cyclotomic \mathbb{Z}_ℓ -extension of L . Then $F(L)$ coincides with the kernel of the map $\pi \circ \theta_L$. If $L = \mathbb{Q}_\ell$ then one can check immediately that $F(\mathbb{Q}_\ell)$ coincides with the kernel of the logarithmic map $\log: \mathbb{Q}_\ell^\times[\ell] \rightarrow \mathbb{Q}_\ell$ (note that the map \log , that was previously defined on \mathbb{Q}_ℓ^\times , can be uniquely extended on $\mathbb{Q}_\ell^\times[\ell]$ by linearity).

If L is a finite extension of \mathbb{Q}_ℓ then the following diagram is commutative

$$\begin{array}{ccc} L^\times[\ell] & \xrightarrow{\theta_L} & G_{\ell,L}^{ab} \\ \downarrow N_{L/\mathbb{Q}_\ell} & & \downarrow res \\ \mathbb{Q}_\ell^\times[\ell] & \xrightarrow{\theta_{\mathbb{Q}_\ell}} & G_{\ell,\mathbb{Q}_\ell}^{ab} \end{array}$$

where N_{L/\mathbb{Q}_ℓ} is the norm map and res the restriction of automorphisms. Thus, $F(L)$ coincides with pre-image of $F(\mathbb{Q}_\ell)$ with respect to the map N_{L/\mathbb{Q}_ℓ} . In other words,

$$F(L) = \{ x \in L^\times[\ell] \mid \text{Sp}_{L/\mathbb{Q}_\ell}(\log x) = 0 \}. \quad (2.16)$$

Put $P(L) = F(L) \cap U^{(1)}(L)$, where $U^{(1)}(L)$ is the group of principal units of L . Then, for any $x \in P(L)$ man has $N_{L/\mathbb{Q}_\ell}(x) = 1$ if $\ell \neq 2$ and $N_{L/\mathbb{Q}_\ell}(x) = \pm 1$ if $\ell = 2$. Hence the group $U^{(1)}(L)$ enters the exact sequence of H -module

$$1 \longrightarrow P(L) \longrightarrow U^{(1)}(L) \xrightarrow{\alpha} \mathbb{Z}_\ell \longrightarrow 0,$$

where $\alpha = \log \circ N_{L/\mathbb{Q}_\ell}$. Thus, using additive notation for the group operation in $P(L)$ and $U^{(1)}(L)$, we get

$$(P(L) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell \cong U^{(1)}(L) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong L \cong \mathbb{Q}_\ell[H]. \quad (2.17)$$

On the other hand, there is an exact sequence of H -modules

$$1 \longrightarrow P(L) \longrightarrow F(L) \xrightarrow{\nu} A \longrightarrow 1, \quad (2.18)$$

where for $x \in F(L)$ by $\nu(x) \in \mathbb{Z}_\ell$ we denote the ℓ -adic exponent of x . The map ν is non-zero since $\ell \in F(L)$ and $\nu(\ell) \neq 0$, therefore A is a subgroup of finite index in \mathbb{Z}_ℓ , hence $A \cong \mathbb{Z}_\ell$. Comparing (2.17) and (2.18), we obtain $F(L) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong U^{(1)}(L) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell[H]$.

Now we return to K . As it was proved before, for any its completion K_v , where $v|\ell$, one has $F(K_v) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell[G_v]$, where G_v is a decomposition subgroup of v in G . Since $(\prod_{v|\ell} F(K_v)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \prod_{v|\ell} (F(K_v) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ and $\prod_{v|\ell} F(K_v) \cong \text{Ind}_G^{G_v} F(K_v)$, we obtain

$$\left(\prod_{v|\ell} F(K_v) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \text{Ind}_G^{G_v} (F(K_v) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \cong \text{Ind}_G^{G_v} \mathbb{Q}_\ell[G_v] \cong R.$$

This proves the lemma.

Now we can easily conclude the proof of the theorem. It follows from (2.2) and (2.5) that

$$(U(K)[\ell] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \oplus (X_0 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell \cong R.$$

On the other hand, by (2.10), (2.15) and Lemma 2.3 we get

$$(U_{S,1}(K) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \oplus (X_0 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \cong R.$$

This proves the theorem.

Consequence. Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and K a Galois extension of \mathbb{Q} with a Galois group G , such that K contains k . Put $H = G(K/k)$ and let τ be an automorphism of complex conjugation in G . In particular, it means that $\tau^2 = 1$ and $\tau(\sqrt{-d}) \neq \sqrt{-d}$. Then in the group $U_{S,1}(K)$ there is an element ω such that $\tau(\omega) = \omega$ and $\{h(\omega)\}_{h \in H}$ generate a submodule of finite index in $U_{S,1}(K)$.

Indeed, in the group of units $U(K)$ there is the so called Artin unit ε such that $\tau(\varepsilon) = \varepsilon$ and $\{h(\varepsilon)\}_{h \in H}$ generate in $U(K)$ a subgroup of finite index. Then $\varepsilon_1 = \varepsilon \oplus 1$ generates an R -module $(U(K) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell$. So the element $\omega_1 = \varphi^{-1}(\varepsilon_1)$, where φ is an isomorphism of Theorem 1, has the property $\tau(\omega_1) = \omega_1$. Then the elements $\{h(\omega_1)\}_{h \in H}$ form the basis of \mathbb{Q}_ℓ -space $U_{S,1}(K) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. If s is sufficiently large then $\ell^s \omega_1 \in U_{S,1}(K)$, so one can put $\omega = \ell^s \omega_1$.

3 Conditional proof of the main conjecture

Firstly, we give more detailed definition of the regulator $\mathfrak{R}_\ell(K)$ for an algebraic number field K and prime ℓ that splits completely in K .

Let $\mathfrak{l}_1, \dots, \mathfrak{l}_m$ run all prime divisors over ℓ in K . Then for any S -unit $u \in U_S(K)$, where $S = \{\mathfrak{l}_1, \dots, \mathfrak{l}_m\}$, there is its principal divisor (u) , which we shall write in additive form as $(u) = \sum_{i=1}^m a_i \mathfrak{l}_i$, where $a_i \in \mathbb{Z}$. We define the group $D(K)$ as a free \mathbb{Z}_ℓ -module with generators $\mathfrak{l}_1, \dots, \mathfrak{l}_m$ and shall consider (u) as an element of $D(K)$. Then to any u from the pro- ℓ -completion $U_S(K)[\ell]$ of the group $U_S(K)$ corresponds its divisor $\text{div}(u) \in D(K)$. If $u = \lim_{n \rightarrow \infty} u_n$, where $u_n \in U_S(K)$, then $\text{div}(u) = \lim_{n \rightarrow \infty} \text{div}(u_n)$. Obviously, $\text{div}(u)$ depends only on the image of u in $\bar{U}_S(K)[\ell] := U_S(K)[\ell] / \mu_\ell(K)$, where $\mu_\ell(K)$ is the group of all roots of unity of ℓ -power degree in K .

We consider on the \mathbb{Z}_ℓ -module $D(K)$ a symmetric non-degenerate bilinear form defined for $x, y \in D(K)$, $x = \sum_{i=1}^m a_i \mathfrak{l}_i$, $y = \sum_{i=1}^m b_i \mathfrak{l}_i$ by

$$\langle x, y \rangle = \sum_{i=1}^m a_i b_i \in \mathbb{Z}_\ell. \quad (3.1)$$

If we wish to stress dependence of the product (3.1) upon K we shall write $\langle x, y \rangle_K$.

If σ is an automorphism of K then σ acts on S and hence on the group $D(K)$. Obviously, one has

$$\langle x, y \rangle = \langle \sigma x, \sigma y \rangle. \quad (3.2)$$

Let K_∞ be the cyclotomic \mathbb{Z}_ℓ -extension of K and $U_{S,1}(K)$ the group of global universal norms defined in the introduction. Let e_1, \dots, e_t be a \mathbb{Z}_ℓ -basis of $U_{S,1}(K)$. Then we put

$$\mathfrak{R}_\ell(K) = \det(\langle \text{div}(e_i), \text{div}(e_j) \rangle), \quad 1 \leq i, j \leq t.$$

If e'_1, \dots, e'_t is another basis of $U_{S,1}(K)$ and C is a transition matrix then the regulators defined by this two basis differs by factor $(\det C)^2 \in \mathbb{Z}_\ell^\times$. In particular, the ℓ -adic exponent $\nu_\ell(\mathfrak{R}_\ell(K))$ does not depend on the choice of the basis in $U_{S,1}(K)$.

Note some simple properties of pairing $\langle \cdot, \cdot \rangle$ and the regulator $\mathfrak{R}_\ell(K)$.

Proposition 3.1. *Suppose that ℓ splits completely in K , and Leopoldt conjecture holds for (K, ℓ) . Then the map $\text{div}: U_{S,1}(K) \rightarrow D(K)$ is an injection.*

Proof. The kernel of the map $\text{div}: \bar{U}_S(K)[\ell] \rightarrow D(K)$ coincides with the pro- ℓ -completion of $\bar{U}(K)[\ell]$, where $\bar{U}_S(K) = U_S(K)/\mu(K)$, $\bar{U}(K) = U(K)/\mu(K)$ and $\mu(K)$ is the group of all roots of unity in K . Thus, it is enough to check that $U_{S,1}(K) \cap (\bar{U}(K)[\ell]) = 1$. If ξ is an element of this intersection, then, as was shown in the proof of Theorem 1, for any injection $\sigma: K \hookrightarrow \mathbb{Q}_\ell$ man has $\log \sigma(\xi) = 0$. It means that ξ^d , where d is an order of $\mu(K)$, belongs to the Leopoldt kernel $E_\ell(K)$, which vanishes by assumption. This proves the proposition.

Let L/K be an extension of algebraic number fields and prime ℓ splits completely in L , hence also in K . Then there are natural maps $i_{L/K}: D(K) \rightarrow D(L)$ and $N_{L/K}: D(L) \rightarrow D(K)$. The first one is induced by the injection $K \hookrightarrow L$ and the second one by the norm map from L^\times to K^\times . Then for any $x, y \in D(K)$ и $z \in D(L)$ man has

$$\langle x, y \rangle_K = [L : K] \langle i_{L/K}(x), i_{L/K}(y) \rangle_L, \quad \langle x, N_{L/K}(z) \rangle_K = \langle i_{L/K}(x), z \rangle_L. \quad (3.3)$$

Indeed, it is enough to check the equalities (3.3) in the case, when x, y, z are prime divisors, but, obviously, in this case they hold true.

Proposition 3.2. *Let L be a Galois extension of \mathbb{Q} with Galois group G , prime ℓ splits completely in L and $K \subseteq L$. Then the condition $\mathfrak{R}_\ell(L) \neq 0$ yields that $\mathfrak{R}_\ell(K) \neq 0$.*

Proof. Put $H = G(L/K)$. The inclusion $K \hookrightarrow L$ induces the inclusion $i: U_{S,1}(K) \hookrightarrow U_{S,1}(L)$, and the norm map $N: L^\times \rightarrow K^\times$ induces the map $N: U_{S,1}(L) \rightarrow U_{S,1}(K)$, whose kernel we denote by B . Since the group $F := i(U_{S,1}(K)) \oplus B$ has finite index in $U_{S,1}(L)$ and $\mathfrak{R}_\ell(L) \neq 0$ by assumption, we obtain that the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate on F . By (3.3) the groups $i(U_{S,1}(K))$ and B are orthogolal with respect to this pairing, hence the pairing

$\langle \cdot, \cdot \rangle_L$ is non-degenerate on the group $i(U_{S,1}(K))$. Then by (3.3) the pairing $\langle \cdot, \cdot \rangle_K$ is non-degenerate on $U_{S,1}(K)$, that is, $\Re_\ell(K) \neq 0$. This proves the proposition.

Though the definition of the regulator $\Re_\ell(K)$ does not use the ℓ -adic logarithms, we need them in order to give conditional proof of the main conjecture (Theorem 2), so we need the following considerations.

Conjecture 5. *Let K be a Galois extension of degree n of \mathbb{Q} with Galois group G , and prime ℓ splits completely in K . Let \mathfrak{l} be a prime divisor of K over ℓ and α such an element of K^\times that $(\alpha) = \mathfrak{l}^{\mathbf{h}}$ for some natural number \mathbf{h} (for example, man can take for \mathbf{h} the class number of K). Then any $n-1$ of the elements $\log_\ell \sigma_1(\alpha), \dots, \log_\ell \sigma_n(\alpha)$, where σ_i runs G , are linearly independent over \mathbb{Q}_ℓ in the space $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^n$.*

Remark. Since $N_{K/\mathbb{Q}}(\alpha) = \ell^h$, one has a relation

$$\sum_{i=1}^n \log_\ell \sigma_i(\alpha) = \log_\ell(N_{K/\mathbb{Q}}(\alpha)) = 0. \quad (3.4)$$

The next result means that Conjecture 5 is in some sense an analog of Leopoldt conjecture. We will not use it in the present paper, but we give it for completeness.

Proposition 3.3. *Let K be an Abelian algebraic number field of degree n and prime ℓ splits completely in K . Then Conjecture 5 holds for K .*

Proof. Put $G = G(K/\mathbb{Q})$, and let $\alpha \in K^\times$ be as in Conjecture 5. Put $\varepsilon_0 = 1 + \ell$ if $\ell \neq 2$ and $\varepsilon_0 = 5$ if $\ell = 2$. Put $\beta = \alpha \varepsilon_0$. Then to prove the proposition, it is enough to check that an $n \times n$ -matrix A , whose lines are the vectors $\log_\ell \sigma_j \sigma_1^{-1}(\beta), \dots, \log_\ell \sigma_j \sigma_n^{-1}(\beta)$, is non-degenerate. A general element of this matrix is of the form $a_{ij} = \log \sigma_j \sigma_i^{-1}(\beta)$. Therefore, the determinant $\det A$ of this matrix is Dedekind determinant, thus by [7, Ch. 3, §6, Theorem 6.1] one has

$$\det A = \prod_{\chi \in \hat{G}} \sum_{\sigma \in G} \bar{\chi}(\sigma) \log \sigma(\beta), \quad (3.5)$$

where \hat{G} is the group of characters of G .

Obviously, the numbers $\sigma_1^{-1}(\beta), \dots, \sigma_n^{-1}(\beta), \ell$ are multiplicatively independent. Hence, according to the ℓ -adic analog of Baker theorem [8], the

numbers $\log \sigma_1^{-1}(\beta), \dots, \log \sigma_n^{-1}(\beta) \in \bar{\mathbb{Q}}_\ell$ are linearly independent over the field of all algebraic numbers $\bar{\mathbb{Q}}$. Therefore, all the factors in the right hand part of (3.5) are non-zero and $\det A \neq 0$. This proves the proposition.

Proposition 3.4. *Conjecture 5 follows from Conjecture 1.*

Proof. Let K/\mathbb{Q} be a Galois extension of degree n with Galois group G and ℓ a prime number that splits completely in K . Let ε_0 and β have the same meaning as in the proof of Proposition 3.3. Reasoning as in the proof of that proposition, we see that it is enough to state non-degenerateness of the $n \times n$ -matrix with a general element $a_{ij} = \log \sigma_j \sigma_i^{-1}(\beta)$, $1 \leq i, j \leq n$. The determinant $\det A$ of this matrix is a polynomial of degree n in n indeterminates $X_1 = \log \sigma_1(\beta), X_2 = \log \sigma_2(\beta), \dots, X_n = \log \sigma_n(\beta)$, $\sigma_i \in G$, where, without restriction of generality, one can assume that $\sigma_1 = 1$. Then the matrix A has the elements X_1 on the main diagonal, and beyond it there are the elements X_2, \dots, X_n . Thus, $\det A = P(X_1, \dots, X_n) = X_1^n + \dots$, where P is a polynomial of degree n in X_1, \dots, X_n and dots mean the sum of monomials that contain X_1 in degree at most $n - 1$. Hence P is a non-zero polynomial. By Conjecture 1 the numbers $\log \sigma_1(\beta), \dots, \log \sigma_n(\beta)$ are algebraically independent, thus $\det A \neq 0$. This proves the proposition.

Together with the group of global universal norms $U_{S,1}(K)$ we consider also the group of local universal norms $U_{S,2}(K)$ defined by

$$U_{S,2}(K) := \{x \in \bar{U}_S(K)[\ell] \mid x \in N_{n,v}(K_{n,v}^\times) \text{ for all } v\}. \quad (3.6)$$

where $N_{n,v}$ is the norm map from completion $K_{n,v}$ of K_n into K_v and v runs all the places of K . Note that $N_{n,v}$ is the norm map with respect to the decomposition subgroup of v . It follows immediately from the definition that $U_{S,2}(K) \supseteq U_{S,1}(K)$.

Note that K_n/K is unramified in any $v \nmid \ell$ and any $u \in U_S(K)$ is a unit outside S . Therefore, the condition (3.6) holds automatically for all $v \nmid \ell$. If $v \mid \ell$ then by (2.16) the condition (3.6) means that $\log(N_{K_v/\mathbb{Q}_\ell}(x)) = 0$. If ℓ splits completely in K then $K_v = \mathbb{Q}_\ell$ for any $v \mid \ell$. Thus (3.6) is equivalent to the condition $\log \sigma(x) = 0$ for any inclusion of K into \mathbb{Q}_ℓ , which, in turn, is equivalent to

$$\log_\ell x = 0. \quad (3.7)$$

Proposition 3.5. *Let K/\mathbb{Q} be a Galois extension of degree n with Galois group G and prime ℓ splits completely in K . Assume that the pair (K, ℓ) satisfies Conjecture 2 (the Leopoldt conjecture) and Conjecture 5. Then the group $U_{S,1}(K)$ has finite index in $U_{S,2}(K)$.*

Proof. By Theorem 1 the \mathbb{Z}_ℓ -rank of $U_{S,1}(K)$ equals $r + 1$, where r is the rank of the group of units $U(K)$, so it is enough to check that the same \mathbb{Z}_ℓ -rank has $U_{S,2}(K)$.

Consider a \mathbb{Q}_ℓ -space of dimension n $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^n$. Let V be a subspace of \mathbb{Q}_ℓ^n generated by $\log_\ell \sigma_i(\alpha)$, where $\sigma_1, \dots, \sigma_n$ runs all inclusions of K in \mathbb{Q}_ℓ and α has the same meaning, as in Conjecture 5. Then any $x \in V$ may be presented as a linear combination

$$x = \sum_{i=1}^n a_i \log_\ell \sigma_i(\alpha), \quad (3.8)$$

where by (3.4) and Conjecture 5 the coefficients $a_i \in \mathbb{Q}_\ell$ are defined uniquely up to a constant summand. In particular, in (3.8) we may assume that $a_n = 0$ and a_1, \dots, a_{n-1} are uniquely defined by x .

Take for x the logarithms of a system of fundamental units of K , that is, $\log_\ell \varepsilon_1, \dots, \log_\ell \varepsilon_r$. Thus, we obtain a system of equalities

$$\sum_{i=1}^{n-1} c_{ij} \log_\ell \sigma_i(\alpha) = \log_\ell \varepsilon_j, \quad 1 \leq j \leq r,$$

where the coefficients $c_{ij} \in \mathbb{Q}_\ell$ are defined uniquely.

Let ℓ^s be such a power of ℓ that $d_{ij} := \ell^s c_{ij} \in \mathbb{Z}_\ell$ for all i, j . Put

$$\eta_j = \prod_{i=1}^{n-1} \sigma_i(\alpha)^{d_{ij}}, \quad 1 \leq j \leq r. \quad (3.9)$$

Then the elements $\eta_j \in U_S(K)[\ell]$ satisfy the condition $\log_\ell \eta_j = 0$, hence the images of the elements η_j in $U_S(K)[\ell]/\mu_\ell(K)$ belong to the group $U_{S,2}(K)$. Now suppose that $y \in U_S(K)[\ell]$ and $y \bmod \mu_\ell(K) \in U_{S,2}(K)$. Then, putting $y_1 = y^{\ell^s}$ for sufficiently large s , we obtain a presentation

$$y_1 = \prod_{i=1}^n \sigma_i(\alpha)^{a_i} \times \prod_{j=1}^r \varepsilon_j^{b_j} \quad (3.10)$$

for some $a_i, b_j \in \mathbb{Z}_\ell$. Since $\ell \in U_{S,2}(K)$, multiplying y_1 by ℓ^{-a_n} and noting that $(\ell) = \mathfrak{l}_1 \cdots \mathfrak{l}_n$, we can assume that $a_n = 0$ in (3.10).

Consider an element $z := y_1 \prod_{j=1}^r \eta_j^{b_j}$. Since $y_1 \in U_{S,2}(K)$, we have $\log_\ell y_1 = 0$, hence $\log_\ell z = 0$. But (3.9) and (3.10) imply that $z = \prod_{i=1}^{n-1} \sigma_i(\alpha)^{h_i}$,

где $h_i = a_i + \sum_{j=1}^r b_j d_{ij}$. By Conjecture 5 the vectors $\log_\ell \sigma_i(\alpha)$, $1 \leq i < n$, are linearly independent over \mathbb{Q}_ℓ . Therefore, $h_i = 0$ for $i = 1, \dots, n-1$, that is, any $y \in U_{S,2}(K)$ has a presentation of the form $y = \ell^{b_0} \prod_{j=1}^r \eta_j^{b_j} \pmod{\mu_\ell(K)}$ for some $b_0, b_1, \dots, b_r \in \mathbb{Q}_\ell$. This proves that the \mathbb{Z}_ℓ -rank of $U_{S,2}(K)$ is at most $r+1$. Then Theorem 1 and the inclusion $U_{S,1}(K) \subseteq U_{S,2}(K)$ yield that the index $(U_{S,2}(K) : U_{S,1}(K))$ is finite. This proves the proposition.

Remark 1. Let K be any (maybe non-Galois over \mathbb{Q}) algebraic number field and prime ℓ splits completely in K . Let L be the Galois closure of K . By Proposition 3.5 one has $(U_{S,2}(L) : U_{S,1}(L)) < \infty$ if the conjectures 2 and 5 hold, but then these conjectures yields that $(U_{S,2}(K) : U_{S,1}(K)) < \infty$.

Remark 2. Let $T_\ell(K_\infty)$ be a Γ -module that enters the proof of Theorem 1. In [3], there was formulated a conjecture (Conjecture 2) that $T_\ell(K_\infty)^\Gamma$ is finite for any algebraic number field K and any prime ℓ . It was stated there (Proposition 7.5) that (in notations of the present paper) $T_\ell(K_\infty)^\Gamma \cong U_{S,2}(K)/U_{S,1}(K)$. Thus, if ℓ splits completely in K then it follows from Proposition 3.5 that the last conjecture is a consequence of Conjecture 1.

Proposition 3.6. *Assume that prime ℓ splits completely in K and Leopoldt conjecture holds for (K, ℓ) . Then the group $U_{S,1}(K)$ has finite index in $U_{S,2}(K)$ if and only if the image of the map $\log_\ell : U_S(K)[\ell] \rightarrow \prod_{v|\ell} K_v$ is of \mathbb{Z}_ℓ -rank $n-1$, where $n = [K : \mathbb{Q}]$.*

Proof. Let $\mathfrak{l}_1, \dots, \mathfrak{l}_n$ be all prime divisors of ℓ in K and $\alpha_1, \dots, \alpha_n \in U_S(K)$ such elements that $(\alpha_i) = \mathfrak{l}_i^{s_i}$ for some natural s_i и $i = 1, 2, \dots, n$. Let $N \subseteq \mathbb{Z}_\ell^n$ be a submodule of all vectors $x = \{x_1, \dots, x_n\}$ such that $\sum_{i=1}^n x_i \log_\ell \alpha_i = \log_\ell u$ for some $u \in U(K)[\ell]$. Then for any $x \in N$ the element $u^{-1} \prod_{i=1}^n \alpha_i^{x_i}$ belongs to $U_{S,2}(K)$. Therefore, the rank of N equals to the rank of $U_{S,2}(K)$. If $U_{S,1}(K)$ has a finite index in $U_{S,2}(K)$ then by Theorem 1 this rank equals $r_1 + r_2$. It means that the module \mathbb{Z}_ℓ^n/N is of the rank r_2 . So, a \mathbb{Z}_ℓ -module $\log_\ell(U_S(K)[\ell])/\log_\ell(U(K)[\ell])$ has the rank r_2 . Since a \mathbb{Z}_ℓ -module $\log_\ell(U(K)[\ell])$ is of the rank $r_1 + r_2 - 1$, we have proved one half of the proposition.

Now we suppose that the \mathbb{Z}_ℓ -rank of $\log_\ell(U_S(K)[\ell])$ is $n-1$. It means that $\log_\ell(U_S(K)[\ell])/\log_\ell(U(K)[\ell])$ has \mathbb{Z}_ℓ -rank r_2 . But then the rank of N is $r_1 + r_2$, that is, the rank of $U_{S,2}(K)$ coincides with the rank of $U_{S,1}(K)$. Hence the index $(U_{S,2}(K) : U_{S,1}(K))$ is finite. This proves the proposition.

Now we can formulate and prove the main result of this section.

Theorem 2. *Let K be an algebraic number field and prime ℓ splits completely in K . Then Conjecture 1 yields that $\mathfrak{R}_\ell(K) \neq 0$.*

Proof. If ℓ splits completely in K then ℓ splits completely in the normal closure L of K . If ℓ splits in an imaginary quadratic field $k = \mathbb{Q}(\sqrt{-d})$ then ℓ splits completely in the field $F = L \cdot k$. By Proposition 3.2 the validity of the theorem for F yields its validity for K .

So, we can assume that K is a Galois extension of \mathbb{Q} with Galois group G and $K \supset k = \mathbb{Q}(\sqrt{-d})$. Let $H = G(K/k)$, $[K : k] = n$ and $\tau \in G$ be an automorphism of complex conjugation. In particular, this means that $\tau^2 = 1$ and $\tau \notin H$. Let $\alpha \in U_S(K)$ be of the same meaning, as in Conjecture 5. By Proposition 3.4 this conjecture holds for K . Hence any $2n - 1$ of the $2n$ elements $\{\log_\ell \sigma(\alpha)\}_{\sigma \in G}$ are linearly independent over \mathbb{Q}_ℓ and form a basis of the \mathbb{Q}_ℓ -space

$$V := \{x \in K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^{2n} \mid \mathrm{Sp}_{K/\mathbb{Q}}(x) = 0\}.$$

Let $\varepsilon \in U(K)$ be an Artin unit, that is, $\tau(\varepsilon) = \varepsilon$ and the collection of n elements $\{\sigma(\varepsilon)\}_{\sigma \in H}$ generates the subgroup of finite index in $U(K)$. Since $\log_\ell(\varepsilon) \in V$, there are coefficients $a_\sigma \in \mathbb{Q}_\ell$, $\sigma \in G$, such that

$$\sum_{\sigma \in G} a_\sigma \log_\ell \sigma(\alpha) = \log_\ell \varepsilon.$$

Note that the coefficients a_σ are defined uniquely up to an arbitrary constant summand. That is, instead of a_σ one can take the coefficients $a'_\sigma = a_\sigma + c$, where $c \in \mathbb{Q}_\ell$ is independent of σ .

Put (in additive notation)

$$z = (1 + \tau) \sum_{\sigma \in G} a_\sigma \sigma(\alpha) \in U_S(K)[\ell].$$

Taking into account that $(1 + \tau)\tau\sigma(\alpha) = (1 + \tau)\sigma(\alpha)$, we obtain

$$z = (1 + \tau) \sum_{h \in H} c_h h(\alpha), \quad c_h = a_h + a_{\tau h}. \quad (3.11)$$

Here the coefficients c_h are defined up to a constant summand and

$$2 \log_\ell \varepsilon = \log_\ell z = (1 + \tau) \sum_{h \in H} c_h \log_\ell h(\alpha). \quad (3.12)$$

Fix an element $h_0 \in H$, $h_0 \neq 1$. Subtracting out of all c_h the coefficient c_{h_0} , we may assume, without restriction of generality, that the coefficient c_{h_0} in (3.12) is zero.

Lemma 3.1. *Suppose that $c_{h_0} = 0$ in (3.12) for some $h_0 \neq 1$. Then, assuming the validity of Conjecture 1, man gets that $n - 1$ coefficients c_h , $h \in H$, $h \neq h_0$, are algebraically independent.*

Proof. Consider a system of $3n - 2$ elements $T = T_1 \cup T_2$, where $T_1 = \{\sigma(\alpha)\}_{\sigma \in G, \sigma \neq 1}$, $T_2 = \{h(\varepsilon)\}_{h \in H, h \neq 1}$. Note that we consider K as a subfield of the field \mathbb{Q}_ℓ , and all logarithms that we introduce below are the elements of this field. The system $T \cup \{\ell\}$ is multiplicatively independent hence the logarithms of the elements of T , that is, $3n - 2$ elements $\log \sigma(\alpha)$, where $\sigma \in G$, $\sigma \neq 1$, and $\log h(\varepsilon)$, where $h \in H$ and $h \neq 1$, are algebraically independent. In other words, the field generated by these logarithms, which we denote by $\mathbb{Q}(\log T)$, is of transcendence degree $3n - 2$ over \mathbb{Q} .

Multiplying both parts of (3.12) by any $\sigma \in H$ and taking into account that the map \log_ℓ is a G -homomorphism, we get, using an additive notation for multiplication,

$$2 \log_\ell \sigma(\varepsilon) = \sum_{h \in H} c_h \log_\ell \sigma(1 + \tau)h(\alpha), \quad c_{h_0} = 0.$$

Equating the first coordinates in this equality of vectors, we obtain

$$2 \log \sigma(\varepsilon) = \sum_{h \in H} c_h \log \sigma(1 + \tau)h(\alpha)$$

for all $\sigma \in H$. Thus, putting $T_3 = \{c_\sigma\}_{\sigma \in H, \sigma \neq h_0}$ (remind that $c_{h_0} = 0$), we obtain that all the elements of $\log T_2$ belong to the field $\mathbb{Q}(\log T_1, T_3)$. Since the transcendence degree of $\mathbb{Q}(\log(T_1 \cup T_2))$ is $3n - 2$, we get that the transcendence degree of $\mathbb{Q}(\log T_1, T_3)$ is at least $3n - 2$. But $T_1 \cup T_3$ is a set in $3n - 2$ elements. Therefore, all the elements of T_3 are algebraically independent. This proves the lemma.

Continue the proof of the theorem. Put $z' = z\varepsilon^{-2}$. Then $\log_\ell z' = 0$, that is, z' satisfy the condition (3.7), hence also the condition (3.6), that is, $z' \in U_{S,2}(K)$. Since we assume the conjecture 1 to be true, we can assume that Leopoldt conjecture and Conjecture 5 hold for K . Then by Proposition 3.5 we have $z_1 := (z')^{\ell^s} \in U_{S,1}(K)$ for some natural s . We shall prove that the matrix $C = (\{h_i(z_1), h_j(z_1)\})$, $1 \leq i, j \leq n$ is non-degenerate, where h_1, \dots, h_n are all the elements of H in some order and $\{ , \}$ means the product (1.3). If we shall prove it then we get that the elements $h_1(z_1), \dots, h_n(z_1)$ are multiplicatively independent over \mathbb{Z}_ℓ (since

their divisors are linearly independent over \mathbb{Q}_ℓ). Thus, by Theorem 1 the elements $h_1(z_1), \dots, h_n(z_1)$ generate a subgroup Y of finite index in $U_{S,1}(K)$. So, the condition $\det C \neq 0$ means that the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate on Y . Then it is non-degenerate on $U_{S,1}(K)$ as well. This will prove the theorem.

To simplify the notations, we shall instead of C to have deal with the matrix $C' := \ell^{-2s} \mathbf{h}^{-2} C$, where \mathbf{h} is defined by condition $(\alpha) = \mathbf{h}$. The general element of this matrix is $\mathbf{h}^{-2} \{h_i(z'), h_j(z')\}$. Taking into account that $\text{div}(z') = \text{div}(z) = \mathbf{h}(1 + \tau) \sum_{\sigma \in H} c_\sigma \sigma(\mathbf{l})$, we obtain

$$\mathbf{h}^{-2} \{h_i(z'), h_j(z')\} = \langle h_i(1 + \tau) \sum_{\sigma \in H} c_\sigma \sigma(\mathbf{l}), h_j(1 + \tau) \sum_{\sigma \in H} c_\sigma \sigma(\mathbf{l}) \rangle. \quad (3.13)$$

Calculating the right hand side of (3.13) and taking into account that

$$\langle \sigma_1(\mathbf{l}), \sigma_2(\mathbf{l}) \rangle = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2, \\ 0 & \text{if } \sigma_1 \neq \sigma_2 \end{cases}$$

for any $\sigma_1, \sigma_2 \in G$, we obtain $\mathbf{h}^{-2} \{h_i(z'), h_j(z')\} = f_{ij}(c_1, \dots, c_{n-1})$, where $f_{ij}(c_1, \dots, c_{n-1})$ is a quadratic form in c_1, \dots, c_{n-1} with coefficients in \mathbb{Q} . Thus, $\det C$ is a polynomial F in c_1, \dots, c_{n-1} of degree $2n$, and we wish to show that $F \neq 0$ as a polynomial. To do this we examine the coefficient at c_1^{2n} in F .

Using bilinearity of the product $\langle \cdot, \cdot \rangle$, we obtain from (3.13) an equality $\mathbf{h}^{-2} \{h_i(z'), h_j(z')\} = A_{ij} + B_{ij}$, where

$$\begin{aligned} A_{ij} &= \sum_{\beta, \gamma \in H} \langle h_i c_\beta \beta(\mathbf{l}), h_j c_\gamma \gamma(\mathbf{l}) \rangle + \sum_{\beta, \gamma \in H} \langle h_i \tau c_\beta \beta(\mathbf{l}), h_j \tau c_\gamma \gamma(\mathbf{l}) \rangle, \\ B_{ij} &= \sum_{\beta, \gamma \in H} \langle h_i \tau c_\beta \beta(\mathbf{l}), h_j c_\gamma \gamma(\mathbf{l}) \rangle + \sum_{\beta, \gamma \in H} \langle h_i c_\beta \beta(\mathbf{l}), h_j \tau c_\gamma \gamma(\mathbf{l}) \rangle. \end{aligned}$$

The elements $h_i \tau \beta$ and $h_j \gamma$ belong to the different classes modulo H . The same statement holds for the elements $h_i \beta$ and $h_j \tau \gamma$. Therefore, $B_{ij} = 0$. Thus, $f_{ij}(c_1, \dots, c_{n-1}) = A_{ij}$.

To calculate A_{ij} , we note that

$$\langle h_i c_\beta \beta(\mathbf{l}), h_j c_\gamma \gamma(\mathbf{l}) \rangle = \begin{cases} c_\beta c_\gamma & \text{if } h_i \beta = h_j \gamma, \\ 0 & \text{if } h_i \beta \neq h_j \gamma. \end{cases} \quad (3.14)$$

$$\langle h_i \tau c_\beta \beta(\mathfrak{l}), h_j \tau c_\gamma \gamma(\mathfrak{l}) \rangle = \begin{cases} c_\beta c_\gamma & \text{if } h_i \tau \beta = h_j \tau \gamma, \\ 0 & \text{if } h_i \tau \beta \neq h_j \tau \gamma. \end{cases} \quad (3.15)$$

On the main diagonal of the matrix C' man has $i = j$, that is $h_i = h_j$, so, it follows from (3.14) and (3.15) that $A_{ii} = \sum_{k=1}^{n-1} 2c_k^2$. Thus, the product of all the elements on the main diagonal of the matrix C' yields into $\det C'$ a summand $\prod_{i=1}^n A_{ii} = (\sum_{k=1}^{n-1} 2c_k^2)^n = 2^n c_1^{2n} + \dots$, where dots mean the sum of monomials, which contain c_1 in powers less than $2n$.

Now consider A_{ij} for $i \neq j$, that is, in the case $h_i \neq h_j$. If this is the case then the equalities $h_i \beta = h_j \gamma$ or $h_i \tau \beta = h_j \tau \gamma$ can take place only for $\beta \neq \gamma$. Then it follows from (3.14) and (3.15) that in the case $i \neq j$ the quadratic form $f_{ij}(c_1, \dots, c_{n-1}) = A_{ij}$ does not contain c_1^2 . Therefore, any summand of $\det C'$, but the product of all the elements on the main diagonal, contains c_1 in power at most $2n - 1$. Thus, $\det C' = F(c_1, \dots, c_{n-1})$ is a non-zero polynomial in c_1, \dots, c_{n-1} with coefficients in \mathbb{Q} . By Lemma 3.1 the elements c_1, \dots, c_{n-1} are algebraically independent over \mathbb{Q} . Therefore, $\det C \neq 0$, and this means that $\Re_\ell(K) \neq 0$. This concludes the proof of the theorem.

4 Proof of Theorem 3

Let K be a finite Abelian extension of an imaginary quadratic $k = \mathbb{Q}(\sqrt{-d})$ and K is Galois over \mathbb{Q} . Put $G = G(K/\mathbb{Q})$, $H = G(K/k)$, and let τ be an automorphism of complex conjugation. In particular, it means that $\tau \in G \setminus H$ and $\tau^2 = 1$. Assume that prime ℓ splits completely in K , and let S be the set of all prime divisors of ℓ in K . Then ℓ splits in k into a product of two prime divisors: $(\ell) = \mathfrak{p}_1 \mathfrak{p}_2$. So, man can present S as a disjointed union $S = S_1 \cup S_2$, where S_i is the set of all prime divisors of K over \mathfrak{p}_i , $i = 1, 2$. Thus, man obtains $D(K) = \mathbb{Z}_\ell[S] = D_1(K) \oplus D_2(K)$, where $D_i(K) = \mathbb{Z}_\ell[S_i]$ for $i = 1, 2$. The group H acts on S_1 and S_2 , and with respect to this action the modules $D_i(K)$ are isomorphic to $\mathbb{Z}_\ell[H]$ for $i = 1, 2$. The automorphism τ acts on $D(K)$, interchanging $D_1(K)$ and $D_2(K)$.

Put $x, y \in D(K)$, $x = \sum_{i=1}^{2n} a_i \mathfrak{l}_i$, $y = \sum_{i=1}^{2n} b_i \mathfrak{l}_i$, where $n = [K : k]$, $\mathfrak{l}_1, \dots, \mathfrak{l}_n \in S_1$, $\mathfrak{l}_{n+1}, \dots, \mathfrak{l}_{2n} \in S_2$. Then, but the pairing $\langle \cdot, \cdot \rangle$ that we have deal with in the preceding section, we define two other pairings: $\langle x, y \rangle_{K,1} = \sum_{i=1}^n a_i b_i$ and $\langle x, y \rangle_{K,2} = \sum_{i=n+1}^{2n} a_i b_i$. We define also a relative pairing

$$\langle x, y \rangle_{K/k} = \langle x, y \rangle_{K,1} \mathfrak{p}_1 + \langle x, y \rangle_{K,2} \mathfrak{p}_2 \in D(k). \quad (4.1)$$

An immediate checking shows that for any $\sigma \in G$ and any $x, y \in D(K)$ one has

$$\langle \sigma(x), \sigma(y) \rangle_{K/k} = \sigma(\langle x, y \rangle_{K/k}), \quad (4.2)$$

where G acts on $D(k)$ by restriction of automorphisms, that is H acts on $D(k)$ identically and τ interchanges \mathfrak{p}_1 and \mathfrak{p}_2 . Obviously, for any $x, y \in D(K)$

$$\langle x, y \rangle_K = \langle x, y \rangle_{K,1} + \langle x, y \rangle_{K,2}. \quad (4.3)$$

Proposition 4.1. *Let $K, k, G, H, \tau, \mathfrak{p}_1, \mathfrak{p}_2$ be as above and $x \in D(K)$ such an element that $\tau(x) = x$. Suppose that τ acts on H by the rule $\tau(h) = \tau h \tau^{-1} = \tau h \tau = h^{-1}$ for any $h \in H$. Then for any $h_i, h_j \in H$ an equality holds*

$$\tau(\langle h_i(x), h_j(x) \rangle_{K/k}) = \langle h_i(x), h_j(x) \rangle_{K/k}. \quad (4.4)$$

In other words,

$$\langle h_i(x), h_j(x) \rangle_1 = \langle h_i(x), h_j(x) \rangle_2. \quad (4.5)$$

If A is a $\mathbb{Z}_\ell[G]$ -module generated by x then for any $x, y \in A$ man has $\tau(\langle x, y \rangle_{K/k}) = \langle x, y \rangle_{K/k}$.

Proof. It follows from (4.2) that

$$\begin{aligned} \tau(\langle h_i(x), h_j(x) \rangle_{K/k}) &= \langle \tau h_i(x), \tau h_j(x) \rangle_{K/k} = \\ &= \langle \tau h_i \tau \tau(x), \tau h_j \tau \tau(x) \rangle_{K/k} = \langle h_i^{-1}(x), h_j^{-1}(x) \rangle_{K/k}. \end{aligned}$$

Applying an automorphism $h_i h_j$ and using (4.2) once more, we get

$$\begin{aligned} \langle h_i^{-1}(x), h_j^{-1}(x) \rangle_{K/k} &= h_i h_j \langle h_i^{-1}(x), h_j^{-1}(x) \rangle_{K/k} = \\ &= \langle h_j(x), h_i(x) \rangle_{K/k} = \langle h_i(x), h_j(x) \rangle_{K/k}. \end{aligned}$$

This proves the proposition.

Let H be a finite Abelian group and \hat{H} the group of one-dimensional characters of H with values in $\bar{\mathbb{Q}}_\ell$. Let Φ be the set of all irreducible characters of H defined over \mathbb{Q}_ℓ . Any character $\varphi \in \Phi$ is a sum of all $\chi \in \hat{H}$, which are conjugate by the action of the Galois Group $G(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. If χ enters such sum we say that χ enters φ or χ divides φ and write down this fact by the formula $\varphi = \sum_{\chi|\varphi} \chi$. If $e_\chi = |H|^{-1} \sum_{h \in H} \bar{\chi}(h)h$ is an idempotent that corresponds χ then the idempotent $e_\varphi = \sum_{\chi|\varphi} e_\chi \in \mathbb{Q}_\ell[H]$ corresponds φ .

Let $A \cong \mathbb{Q}_\ell[H]$ be a free $\mathbb{Q}_\ell[H]$ -module of rank 1. Then $A \cong \bigoplus_{\varphi \in \Phi} A_\varphi$, where $A_\varphi = e_\varphi A$. For any $\varphi \in \Phi$ $A_\varphi \neq 0$ and A_φ is an irreducible $\mathbb{Q}_\ell[H]$ -module. Note that for $\varphi \neq \varphi_1$ one has $A_\varphi \not\cong A_{\varphi_1}$. For a character $\varphi \in \Phi$, where $\varphi = \sum_{\chi|\varphi} \chi$, by $\bar{\varphi}$ we denote the character $\sum_{\chi|\varphi} \chi^{-1}$. Obviously, if φ is irreducible then so is $\bar{\varphi}$ and vice versa.

Proposition 4.2. *Let A be a free $\mathbb{Q}_\ell[H]$ -module of rank 1, which carries a non-degenerate symmetric bilinear pairing $(\ , \) : A \times A \rightarrow \mathbb{Q}_\ell$, which satisfies the condition $(ha, hb) = (a, b)$ for any $a, b \in A$, $h \in H$. Then the pairing $(\ , \) : A_\varphi \times A_{\bar{\varphi}} \rightarrow \mathbb{Q}_\ell$ is non-degenerate, while the pairing $(\ , \) : A_\varphi \times A_\psi \rightarrow \mathbb{Q}_\ell$ vanishes for any $\varphi \in \Phi$ and $\psi \neq \bar{\varphi}$.*

Proof. Let m be an exponent of H and $L = \mathbb{Q}_\ell(\zeta_m)$, where ζ_m is a primitive root of unity of degree m . Then any character $\chi \in \hat{H}$ is defined over L and the $L[H]$ -module $B = A \otimes_{\mathbb{Q}_\ell} L$ decomposes over L into one-dimensional components $B = \bigoplus_{\chi} B_\chi$, where $B_\chi = e_\chi B$. Here $A_\varphi \otimes_{\mathbb{Q}_\ell} L = \bigoplus_{\chi|\varphi} B_\chi$ for any $\varphi \in \Phi$. The pairing $(\ , \)$ induces L -bilinear pairing $(\ , \)_1 : B \times B \rightarrow L$, which again satisfies condition $(a, b)_1 = (ha, hb)_1$ for any $a, b \in B$, $h \in H$. If $a, b \in A$ then $(a, b) = (a, b)_1$. Consider the pairing $(\ , \)_1 : B_\chi \times B_\theta \rightarrow L$ for two characters $\chi, \theta \in \hat{H}$. Since $he_\chi = \chi(h)e_\chi$ for any $h \in H$, $\chi \in \hat{H}$, we get

$$(e_\chi a, e_\theta b)_1 = (he_\chi a, he_\theta b)_1 = (\chi(h)e_\chi a, \theta(h)e_\theta b)_1 = \chi\theta(h)(e_\chi a, e_\theta b)_1$$

for any $a, b \in B$, $h \in H$.

Thus, if $\chi\theta \neq 1$ then $(B_\chi, B_\theta)_1 = 0$. If $\psi \neq \bar{\varphi}$ then for any $\chi|\varphi, \theta|\psi$ one has $\chi\theta \neq 1$, that is, the components B_φ and B_ψ are mutually orthogonal. Therefore A_φ, A_ψ are also orthogonal with respect to the pairing $(\ , \)$. The pairing $A_\varphi \times A_{\bar{\varphi}}$ must be non-degenerate, since otherwise it should be zero because of irreducibility of A_φ . In such a case the pairing $A_\varphi \times A$ should be zero also, but this contradicts the conditions of the proposition. This proves the proposition.

Theorem 3. *Let K be an Abelian extension of an imaginary quadratic field $k = \mathbb{Q}(\sqrt{-d})$ with Galois group $H = G(K/k)$. Suppose that K is Galois over \mathbb{Q} and prime ℓ splits completely in K . Let the complex conjugation $\tau \in G = G(K/\mathbb{Q})$ acts on H by the rule $\tau(h) = \tau h \tau^{-1} = h^{-1}$ for any $h \in H$. Let the exponent m of H be such that the primitive root of unity $\zeta_m \in \bar{\mathbb{Q}}_\ell$ of degree m is conjugated with ζ_m^{-1} by the action of the Galois group $G(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Then we have $\Re_\ell(K) \neq 0$.*

Proof. By the consequence of Theorem 1 there is an element $\omega \in U_{S,1}(K)$ such that $\tau(\omega) = \omega$ and the elements $\{h(\omega)\}_{h \in H}$ generate in $U_{S,1}(K)$ a \mathbb{Z}_ℓ -submodule of finite index, which we denote by P . By virtue of Theorem 1 P is of \mathbb{Z}_ℓ -rank $r + 1 = |H| = n$, where r is the rank of the group of units $U(K)$ of K . Therefore $P \cong \mathbb{Z}_\ell[H]$ as $\mathbb{Z}_\ell[H]$ -module. Note that any $g \in G \setminus H$ is of the form $g = h\tau$ for some $h \in H$ hence $g(\omega) = h(\omega)$, that is, P is also a G -module.

Since the Leopoldt conjecture is valid in K , by Prop. 3.1 the natural map $\text{div}: P \rightarrow D(K)$ is an injection. This map induces two homomorphisms: $\text{div}_1: P \rightarrow D_1(K)$ and $\text{div}_2: P \rightarrow D_2(K)$, so that $\text{div} = \text{div}_1 \oplus \text{div}_2$.

For technical convenience we pass from \mathbb{Z}_ℓ -modules to \mathbb{Q}_ℓ -spaces. Put $\hat{P} = P \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We define $\hat{D}(K)$, $\hat{D}_1(K)$ and $\hat{D}_2(K)$ in analogous way. Then the above-mentioned maps div, div_1 and div_2 induce the maps, which we denote by the same symbols, $\text{div}: \hat{P} \rightarrow \hat{D}(K)$, $\text{div}_1: \hat{P} \rightarrow \hat{D}_1(K)$, $\text{div}_2: \hat{P} \rightarrow \hat{D}_2(K)$. The pairing $\langle \cdot, \cdot \rangle_K$ extends uniquely to the pairing $\hat{D}(K) \times \hat{D}(K) \rightarrow \mathbb{Q}_\ell$, which we also denote by $\langle \cdot, \cdot \rangle_K$. Analogously, we have the pairings $\langle \cdot, \cdot \rangle_{K/k}: \hat{D}(K) \times \hat{D}(K) \rightarrow \hat{D}(k)$ and

$$\langle \cdot, \cdot \rangle_{K,1}: \hat{D}_1(K) \times \hat{D}_1(K) \rightarrow \mathbb{Q}_\ell, \quad \langle \cdot, \cdot \rangle_{K,2}: \hat{D}_2(K) \times \hat{D}_2(K) \rightarrow \mathbb{Q}_\ell.$$

To prove the theorem, we have to check that the pairing $\langle \cdot, \cdot \rangle_K$, being restricted to $\text{div}(\hat{P})$, induces a non-degenerate pairing

$$\langle \cdot, \cdot \rangle_K: \text{div}(\hat{P}) \times \text{div}(\hat{P}) \rightarrow \mathbb{Q}_\ell. \quad (4.6)$$

Let Φ be the set of all \mathbb{Q}_ℓ -irreducible characters of H . Suppose that the pairing (4.6) has a non-trivial left kernel, for example. Then this kernel has to contain the component $\text{div}(\hat{P}_\varphi)$ for some $\varphi \in \Phi$, that is, man has $\text{div}(\hat{P}_\varphi) \times \text{div}(\hat{P}) = 0$. On the other hand, since div is an injection, at least one of two conditions $\text{div}_1(\hat{P}_\varphi) \neq 0$ or $\text{div}_2(\hat{P}_\varphi) \neq 0$ holds. Suppose $\text{div}_1(\hat{P}_\varphi) \neq 0$. Since both $\hat{D}_1(K)$ and \hat{P} are isomorphic to $\mathbb{Q}_\ell[H]$, we obtain $\text{div}_1(\hat{P}_\varphi) = \hat{D}_1(K)_\varphi$. By the conditions of the theorem $\varphi = \bar{\varphi}$, hence by Proposition 4.2 the non-degenerate pairing $\langle \cdot, \cdot \rangle_{K,1}$ induces a non-degenerate pairing $\hat{D}_1(K)_\varphi \times \hat{D}_1(K)_\varphi \rightarrow \mathbb{Q}_\ell$. Therefore the pairing

$$\langle \cdot, \cdot \rangle_{K,1}: \text{div}(\hat{P}_\varphi) \times \text{div}(\hat{P}_\varphi) \rightarrow \mathbb{Q}_\ell \quad (4.7)$$

is also non-degenerate.

By the consequence of Theorem 1 the module P satisfies the assertion of Proposition 4.1, that is, for any $a, b \in P$ we have $\langle a, b \rangle_{K,1} = \langle a, b \rangle_{K,2}$. Thus, for any $a, b \in \hat{P}_\varphi$, one has the equality $\langle \text{div}(a), \text{div}(b) \rangle_K = 2\langle \text{div}(a), \text{div}(b) \rangle_{K,1}$. So the non-degeneracy of (4.7) yields the non-degeneracy of the pairing $\langle \cdot, \cdot \rangle_K: \text{div}(\hat{P}_\varphi) \times \text{div}(\hat{P}_\varphi) \rightarrow \mathbb{Q}_\ell$, which contradicts the assumption that $\text{div}(\hat{P}_\varphi)$ enters the kernel of (4.6). The case $\text{div}_2(\hat{P}_\varphi) \neq 0$ can be treated analogously. This proves the theorem.

We assume in Theorem 3 a primitive root of unity ζ_m of degree m to be conjugated with ζ_m^{-1} in the field \mathbb{Q}_ℓ , where m is the exponent of H . Obviously, the validity of this condition depends only on ℓ and m . The next proposition characterize all pairs (ℓ, m) , which have this property.

Proposition 4.3. *Let ℓ be a prime number and m an arbitrary natural number. If $\ell \neq 2$ then we write m in the form $m = 2^a \ell^b c$ (if $\ell = 2$ in the form $m = 2^a c$, where c is prime to 2ℓ). Let c_1 be the product of all different p that divide c .*

The root $\zeta_m \in \bar{\mathbb{Q}}_\ell$ is conjugated with ζ_m^{-1} over \mathbb{Q}_ℓ if and only if one of the following conditions holds:

- (i) $\ell \neq 2$, $a = 0, 1$ and there is a natural number r such that $\ell^r + 1 \equiv 0 \pmod{c_1}$;
- (ii) $\ell \equiv 3 \pmod{4}$, $a \geq 2$ and there is an odd natural number r such that $\ell^r + 1 \equiv 0 \pmod{2^a c_1}$;
- (iii) $\ell = 2$ and there is a natural number r such that $\ell^r + 1 \equiv 0 \pmod{c_1}$.

Proof. Suppose that there is $\sigma \in G(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ such that for given m one has $\sigma(\zeta_m) = \zeta_m^{-1}$. For $m = 1, 2$ we can assume σ to be an identity automorphism. For $m > 2$, the condition $\sigma(\zeta_m) = \zeta_m^{-1}$ means that σ is an automorphism of order two of the field $F = \mathbb{Q}_\ell(\zeta_m)$.

At first, suppose that $\ell \neq 2$. The field F is a free composite over \mathbb{Q}_ℓ of $F_1 = \mathbb{Q}_\ell(\zeta_{\ell^b})$ and $F_2 = \mathbb{Q}_\ell(\zeta_{2^a c})$. The extension F_1/\mathbb{Q}_ℓ is purely ramified and has a Galois group isomorphic to $(\mathbb{Z}_\ell/\ell^b \mathbb{Z}_\ell)^\times$. The automorphism σ_1 , which corresponds to -1 under the last isomorphism, sends ζ_{ℓ^b} into $\zeta_{\ell^b}^{-1}$ and fixes any root of unity of degree prime to ℓ . The extension F_2/\mathbb{Q}_ℓ is unramified, hence have a cyclic Galois group. Suppose that in $G(F_2/\mathbb{Q}_\ell)$ there is an automorphism σ_2 such that $\sigma_2(\zeta_{2^a c}) = \zeta_{2^a c}^{-1}$. Then the automorphism $\sigma =$

$\sigma_1\sigma_2$ satisfies the condition $\sigma(\zeta_m) = \zeta_m^{-1}$. Thus, if $\ell \neq 2$ then ζ_m and ζ_m^{-1} are conjugated if and only if the roots $\zeta_{2^a c}$ and $\zeta_{2^a c}^{-1}$ are conjugated in F_2 .

1. If $a = 0, 1$ then $\zeta_{2^a c}$ and $\zeta_{2^a c}^{-1}$ are conjugated if and only if ζ_c and ζ_c^{-1} are conjugated by some σ_2 . Let p be a prime divisor of c . Let $E = F_2^{\sigma_2}$ and \bar{E}, \bar{F}_2 be residue fields of E and F_2 respectively. If $|\bar{E}| = \ell^r$ then $|\bar{E}^\times| = \ell^r - 1$ and $|\bar{F}_2^\times| = \ell^{2r} - 1 = (\ell^r - 1)(\ell^r + 1)$. If $\zeta_p \in F_2$ then $|\bar{F}_2^\times| \equiv 0 \pmod{p}$ but man has $\sigma_2(\zeta_p) = \zeta_p^{-1}$. It means that $\zeta_p \notin E$, that is, $\ell^r - 1 \not\equiv 0 \pmod{p}$. Since p is an arbitrary prime divisor of c , we obtain $\ell^r + 1 \equiv 0 \pmod{c_1}$.

On the contrary, assume that $\ell^r + 1 \equiv 0 \pmod{c_1}$ and $a = 0, 1$. Take for E the unramified extension of \mathbb{Q}_ℓ of degree r , and let E_2 be the unique quadratic unramified extension of E . The condition $\ell^r + 1 \equiv 0 \pmod{c_1}$ implies that $\ell^r - 1$ is prime to c_1 , that is, $\zeta_{c_1} \in E_2$ and $\zeta_p \notin E$ for any $p|c$. Since $\zeta_p \in E_2$, man can obtain the field $F_2 = E_2(\zeta_c)$ as follows. We already have $\zeta_{c_1} \in E_2$. If $c_1 = c$ then man can put $F_2 = E_2$. If $c_1 \neq c$ then there is $p|c$ such that $c_2 = pc_1|c$. Put $E'_2 = E_2(\sqrt[p]{\zeta_{c_1}})$. Then either $E'_2 = E_2$ or E'_2/E_2 is a Kummer extension and hence has degree p . If $c_2 \neq c$ then c has a divisor $c_3 = qc_2$, where q is a prime divisor of c (possibly, $q = p$). Then either the field $E''_2 := E'_2(\sqrt[q]{\zeta_{c_2}})$ coincides with E'_2 or it is an extension of degree q of the latter.

Continuing in such a way, we get some extension L of E_2 such that L contains ζ_c and L has odd degree over E_2 . Therefore $[L : E] = 2f$ for some odd f , and there is a subfield $L' \subset L$ such that L is quadratic over L' . In particular, this means that L is an extension of E_2 of some odd degree f . Let σ_2 be the unique non-trivial automorphism of L/L' . Since L' has odd degree over E and for any $p|\ell$ the primitive root of unity ζ_p belongs to the quadratic extension E_2 of E , we obtain that L' has no roots ζ_p for $p|c$. It means that $N_{L/L'}(\zeta_c) = 1$ or, in other words $\sigma_2(\zeta_c) = \zeta_c^{-1}$. Then, putting $F_2 = L$ and $E = L'$, we get the desired.

2. Now we assume that $\ell \neq 2$ and $a \geq 2$. If $\ell \equiv 1 \pmod{4}$ then the field \mathbb{Q}_ℓ contains a primitive root of unity ζ_4 of degree 4. If ζ_m would be conjugated with ζ_m^{-1} then ζ_4 should be conjugated with ζ_4^{-1} . But it is impossible because of the condition $\zeta_4 \in \mathbb{Q}_\ell$. Thus in the case $a \geq 2$ the roots ζ_m and ζ_m^{-1} can be conjugated only if $\ell \equiv 3 \pmod{4}$. Moreover, if σ_2 is an automorphism of order two of $F_2 = \mathbb{Q}_\ell(\zeta_{2^a c})$ then $\sigma_2(\zeta_4) = \zeta_4^{-1}$, that is, $\zeta_4 \notin E = F_2^{\sigma_2}$. It means that $\ell^r - 1 \equiv 2 \pmod{4}$, where $r = [E : \mathbb{Q}_\ell]$, that is, E is an extension of \mathbb{Q}_ℓ of odd degree. Just as in the paragraph 1, we obtain the congruence $\ell^r + 1 \equiv 0 \pmod{c_1}$.

Let 2^s be the maximal degree of 2-power that contains in F_2 . Note that

s is the 2-adic exponent of $\ell^{2^r} - 1$. As it was shown above, the 2-component of the group of the roots of unity in E consists of ± 1 . On the other hand, since the extension F_2/E is unramified, the norm map $N: F_2^\times \rightarrow E^\times$ induces a surjection of the multiplicative groups of the residue fields. It means that $N(\zeta_{2^s}) = -1$, that is, $\sigma_2(\zeta_{2^s}) = -\zeta_{2^s}^{-1}$. Thus the condition $\sigma_2(\zeta_{2^a}) = \zeta_{2^a}^{-1}$ holds only if $s > a$. This inequality is equivalent to the condition $\ell^r + 1 \equiv 0 \pmod{2^a}$. This proves the necessity of the condition formulated in paragraph 2. Man can prove its sufficiency, reasoning as in paragraph 1.

3. Now suppose $\ell = 2$. In this case $m = 2^a c$, where $(2, c) = 1$. We have to construct an unramified extension $F_2 = \mathbb{Q}_2(\zeta_c)/\mathbb{Q}_2$ and an automorphism of order two σ_2 of the field F_2 such that $\sigma_2(\zeta_c) = \zeta_c^{-1}$. The same reasoning, as we used in paragraph 1, shows that such σ_2 exists if and only if $\ell^r + 1 \equiv 0 \pmod{c_1}$ for some r . This proves the proposition.

Examples. 1. If $\ell = 5$ then $\ell^3 + 1 = 126 = 2 \cdot 3^2 \cdot 7$, hence any m of the form $m = 2^a 3^{s_1} 5^{s_2} 7^{s_3}$, where $a = 0, 1$, satisfies condition (i) of Prop. 4.3. Therefore, for $\ell = 5$ and any K such that the Galois group $H = G(K/k)$ is of exponent m of the above mentioned form, man has $\mathfrak{R}_5(K) \neq 0$ provided 5 splits completely in K .

2. Put $\ell = 11$. Then $\zeta_5 \in \mathbb{Q}_{11}$ hence Theorem 3 does not hold for any m that divides by 5.

Now we assume that the field K from Theorem 3 is fixed. Our goal is to characterize all prime ℓ such that Theorem 3 holds for the pair (K, ℓ) .

Proposition 4.4. *Let a field K be Galois over \mathbb{Q} and K is an Abelian extension of some imaginary quadratic field k , as in Theorem 3. Let m be an exponent of $H = G(K/k)$ and T the set of all primes that do not divide m . Put $F = \mathbb{Q}(\zeta_m)$.*

If $E := K \cap F$ is totally real (in particular, if $E = \mathbb{Q}$) then there are infinitely many primes $\ell \in T$, which satisfy all conditions of Theorem 3. In particular, man has $\mathfrak{R}_\ell(K) \neq 0$ for these ℓ . Otherwise, the conditions of Theorem 3 take place for no $\ell \in T$.

Proof. Put $L = K \cdot F$. Let δ be the automorphism of complex conjugation in F . If the field E is totally real then δ acts on E as identity. Since L is a free composite of K and F over E , there is an automorphism $\sigma \in G(L/E)$ such that $\sigma|_K = \text{id}$ and $\sigma|_F = \delta$. By the Chebotarev density theorem there are infinitely many prime divisors \mathfrak{l} of L such that \mathfrak{l} is unramified in L and the Frobenius automorphism $\varphi(\mathfrak{l})$ equals σ . Let ℓ be a prime rational number under \mathfrak{l} . Since σ induces the identity automorphism of K , we see

that ℓ splits completely in K ; while the fact that σ and δ coincide, being restricted to the cyclotomic field $F = \mathbb{Q}(\zeta_m)$, means that $\ell \equiv -1 \pmod{m}$. By Prop. 4.3 any such ℓ satisfies conditions of Theorem 3 (with $r = 1$), hence for such ℓ one has $\Re_\ell(K) \neq 0$.

On the contrary, assume that E is an imaginary field. Suppose that for some prime $\ell \in T$ the roots ζ_m and ζ_m^{-1} are conjugated over \mathbb{Q}_ℓ . Then by Prop. 4.3 one has a congruence $\ell^r + 1 \equiv 0 \pmod{m}$ for some r . Let $\varphi(\mathfrak{l})$ be the Frobenius automorphism corresponding to ℓ . Then the last congruence means that $\varphi(\mathfrak{l})^r$ coincides with δ on F . Since E is imaginary, δ acts non-trivially on E , hence $\varphi(\mathfrak{l})$ acts non-trivially on K , that is, ℓ does not split completely in K . This proves the proposition.

Remark. Let $p \equiv 3 \pmod{4}$ be a prime number, $k = \mathbb{Q}(\sqrt{-p})$ and K an Abelian extension of k , as in Theorem 3, such that the group $H = G(K/k)$ is of exponent p . In this case we have $F = \mathbb{Q}(\zeta_p)$ and $F \cap K \supset k$, so by Prop. 4.4 the assumptions of Theorem 3 don't hold for any prime ℓ but p .

Nevertheless if $\ell = p$ one can apply this theorem and obtain $\Re_\ell(K) \neq 0$.

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L. V. Kuz'min
 NRC “Kurchatovskii institute”
 E-mail: helltiapa@mail.ru